BUILDING THEORIES: THE THREE WORLDS OF MATHEMATICS

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A comment on 'Three worlds and the imaginary sphere', Inglis, 23(3): Even though a theory of 'three worlds' has been developing for just over two years, it is still under construction. A theory in progress is a particularly delicate creation. Theories do not appear fully formed. There is a period of exploration and incubation that precedes the eventual formulation. In the case of the theory presented here, although it is moving towards a stable form, it is still in the process of being filled out and refined I say this, not to excuse errors of omission or obfuscation, but to be honest about the current stage of development and to encourage others to reflect on the ideas to help in filling them out. We build theories by reflecting on our experience and, because we have different experiences, we naturally produce different theories or different aspects of a theory that can be made stronger by refinement.

Theory building needs to nurture surprising insights and allow them to grow Aggressive criticism comes later Mason *et al.*(1982) talked about three different kinds of specialization (see p 25) in their book *Thinking mathematically*:

- *random specialization* - in which one tries a few examples almost at random to get an idea of what is going on
- *systematic specialization* - in which one considers particular cases in a more organized way to build a theory
- *artful specialization* - in which one chooses cunning cases that test the theory.

The development of the theory of 'three worlds of mathematics' has gone beyond random specialization and on to systematic specialization in three areas: calculus (Tall, 2003), proof (Tall, 2002b) and vectors (Watson, Spyrou and Tall, 2002)

In his communication, 'Three worlds and the imaginary sphere', Matthew Inglis (2003) has published an "artful specialization" to test the theory of 'three worlds of mathematics' based on the evidence of these three systematic specialisations. It is a criticism of a growing theory that can usefully test the theory and either destroy it, or, as I shall shortly show, make it stronger

A history of the theory

For several years I have been working with Eddie Gray and others on the ways in which we conceptualize different kinds of mathematical concept (Tall, 1995; Gray *et al.*, 1999; Tall *et al.*, 2000a; Tall *et al.*, 2000b). Eddie and I were particularly interested in the distinction between objects formed in geometry (such as points, lines, circles and polyhedra) and concepts studied in arithmetic, algebra and symbolic calculus (numbers, algebraic expressions and limits).

We concluded that the development of geometric concepts followed a natural growth of sophistication ably described by van Hiele (1986) in which objects were first perceived as whole gestalts, then roughly described, with language growing more sophisticated so that descriptions became definitions suitable for deduction and proof. However, numbers and algebra began through compressing the process of counting to the concept of number and grew in sophistication through the development of successive concepts where processes were symbolised and used dually as concepts (sum, product, exponent, algebraic expression as evaluation and manipulable concept, limit as potentially infinite process of approximation and finite concept of limit).

We were also intrigued by the way in which experiences in elementary mathematics were re-conceptualised from concepts that necessarily *had* properties, to the formalism of advanced mathematics where specified *properties* are stated first as axioms and definitions, then other properties are deduced by formal proof.

In Gray and Tall (2001), we presented the idea that there were three (or possibly four) fundamentally different types of object:

- those that arise through *empirical abstraction* (in the sense of Piaget) by which is meant the study of *objects* to discover their properties
- those that arise from what Piaget termed *pseudo-empirical abstraction* from focusing on *actions* (such as counting) that are symbolised and mentally compressed as *concepts* (such as number)
- those that arise from the study of *properties*, and the logical deductions that follow from these, found in the modern formalist approach to mathematics.

Piaget also formulated the notion of *reflective abstraction* (which is essentially a more sophisticated version of pseudo-empirical abstraction) in which the focus is on actions on mental objects which are routinized, then conceptualised as processes and considered as mental objects at a higher level.

Our possible 'fourth type of concept' described in Gray and Tall (2001) arose from distinguishing between abstract versions of pseudo-empirical abstraction (focusing on actions on mental objects) and abstract versions of empirical abstraction (focusing on the properties of abstract mental concepts) This could be considered as distinguishing between formal generalizations of arithmetic, algebra and symbolic calculus to give subjects like algebraic number theory, groups, rings, fields, vector spaces and analysis and formal generalizations of geometric objects to give non-Euclidean geometries. However, because all types of formal mathematics involve specifying a system of formal axioms for a type of axiomatic structure and deducing the properties of that structure by formal proof, we settled on categorising all such formal mathematics under a single heading Such a formulation was not set in stone. We were very open to suggestions and criticism to test and improve our ideas

It was during the expansion of these ideas that I worked with Anna Poynter who, as Anna Watson before her recent marriage, was researching students' conceptualisation of vectors. Two papers (Watson, 2002; Watson *et al.*, 2002) reveal the nature of our deliberations and are the first published indications of a developing theory of three mathematical worlds Her study revealed two different kinds of approaches to vectors in school:

- *geometric* - with vectors as arrows representing various embodied concepts such as force, journey, velocity and acceleration
- *symbolic* - based on calculation with matrices

and these were contrasted with a *formal* approach introduced at university level based on deduction from the axioms of a vector space We realised that there were not only three distinct types of mathematical concept (geometric, symbolic and axiomatic), there were actually three very different types of cognitive development which inhabited three distinct mathematical worlds.

The *first* grows out of our *perceptions* of the world and consists of our thinking about things that we perceive and sense, not only in the physical world, but in our own mental world of meaning By reflection and by the use of increasingly sophisticated language, we can focus on aspects of our sensory experience that enable us to envisage conceptions that no longer exist in the world outside, such as a 'line' that is 'perfectly straight' I now term this world the 'conceptual-embodied world' or '*embodied* world' for short

This is not the same as the notion of 'embodiment' in authors such as Lakoff, who focuses on all kinds of embodiment, including *conceptual* - which refers to conceiving concepts in visuo-spatial ways - and *functional* - in terms of the (possibly unconscious) ways of operating using human abilities as biological individuals. Lakoff and his colleagues assert, in their own broad meaning, that *everything is embodied* (Lakoff and Johnson, 1999; Lakoff and Nunez, 2000). This is fine to make a point (that mathematics arises from biological human activity) but a classification with only one class is hardly helpful to analyse the nature of mathematical cognition Instead I focus more on the notion of conceptual embodiment, which relates to the way in which we build more sophisticated notions from sensory experiences

By formulating the embodied world in this way, it includes not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuo-spatial imagery. It therefore applies not only to the conceptual development of Euclidean geometry, but also other geometries that can be conceptually embodied such as non-Euclidean geometries that can be imagined visuo-spatially on surfaces other than flat Euclidean planes and any other mathematical concept that is conceived in visuo-spatial and other sensory ways.

The *second* world is the world of symbols that we use for calculation and manipulation in arithmetic, algebra, calculus and so on. These begin with *actions* (such as pointing and counting) that are encapsulated as concepts by using symbols that allow us to switch effortlessly from processes to *do* mathematics, to concepts to *think* about In collaboration with Eddie Gray, we had realised that symbols such as 3 + 2 in arithmetic had dual connotations as process (addition) and concept (sum). The phenomenon by which a symbol can enable us to switch fluently from processes to do and concepts to think about was enshrined in the formulation of the term "procept" (Gray and Tall, 1994) This second world I call the 'proceptual-symbolic world' or simply the '*proceptual* world'

The notion of procept initially builds on actions in the embodied world. The initial stages of counting and early arithmetic are largely embodied But the focus on the properties of the symbols and the relationship between them moves away from the physical meaning to a symbolic activity in arithmetic. This becomes increasingly sophisticated, with the introduction of more sophisticated number concepts (fractions, negatives, rationals, irrationals, infinite decimals, complex numbers, vectors in two and three, then *n* dimensions, and so on) that enable us to calculate and manipulate symbols with great accuracy and precision. It moves on into generalised arithmetic and algebra through the manipulation of symbols to specify and solve equations, and to more general concepts in symbolic calculus and beyond

This suggests that many symbolic concepts arise from natural embodiments, and lead on to more sophisticated symbolism In fact, there are many occasions when individuals do not encapsulate a given process into a thinkable mental object and instead carry out the procedures in a routinized way based on repetition and interiorization of learned operations. This happens not only with students who fail, it can happen in a very successful way, in which familiar procedures are performed on symbols that do not have natural conceptual embodiments for the individual concerned.

For instance, in solving cubic equations in the sixteenth century, Tartaglia and Cardano (in his *Ars Magna*, 1545) performed calculations that led to the square roots of negative numbers that happened to cancel and, in the end, give a genuine real solution. Such 'numbers' were initially devoid of any link to the individual's geometric imagination Nowadays, we are presented with complex numbers embodied as points in the plane. This shows that, over time, using symbol manipulation (which is, of course, functionally embodied because it is performed by imagining we are moving symbols around) can lead, in the end, back to a meaningful conceptual embodiment.

The third world is based on *properties*, expressed in terms of formal definitions that are used as axioms to specify mathematical structures (such as 'group', 'field', 'vector space', 'topological space' and so on). This world I have called the 'formal-axiomatic world' or '*formal* world', for short It turns previous experiences on their heads, working not with familiar objects of experience, but with axioms that are carefully formulated to *define* mathematical structures in terms of specified properties. Other properties are then deduced by formal proof to build a sequence of theorems Within the axiomatic system, new concepts can be defined and their properties deduced to build a coherent, logically deduced theory.

The formal world arises from a combination of embodied conceptions and symbolic manipulation, but the reverse can, and does, happen. Formal definitions and formal deductions can lead to special theorems called 'structure theorems'. These show that a formal axiomatic system can be proved to have properties that give it a new, more sophisticated embodiment

For instance, an axiomatic group can be embodied through Cayley's theorem as a subgroup of a group of permutations, returning formal group theory to the embodied idea of permuting elements of a set A finite-dimensional vector space is structurally isomorphic to a space of n-tuples, wherein two- and three- dimensional vector spaces over **R** are just two-dimensional and three-dimensional space, with higher dimensions easily computed symbolically but less easily imagined visually The definition of a complete ordered field can be proved to be unique, and therefore embodied by the visual conception of a number line 'completed' by adding the irrational numbers to the real line.

However, even these structure theorems may be embodied in ways that have incidental properties that suggest theorems that turn out not to be true. For instance, the 'completion' of the rational numbers, adding the irrationals to give the real number line, is often conceived as being the ultimate destination, with the real numbers filling out the whole line, banishing the possibility of infinitesimal quantities on the line Yet this conceptualization limits our imagination and is simply untrue in the formal world of mathematics. It is very simple, mathematically, to place the ordered field **R** in a larger ordered field (e g the field of rational functions consisting of quotients of polynomials in an indeterminate x) which can be imagined mentally as a more sophisticated line that can be magnified to 'see' infinitesimal quantities (Tall, 2002a).

The theory of 'three worlds of mathematics' has ramifications that continue to be the focus of careful consideration and long-term reflection.

Current developments

At this point I must give credit to Anna Poynter, who is the only person who has published an article with me on 'three worlds of mathematics' and who was instrumental in inspiring my own ideas on this topic Inglis (2003) refers to "Gray and Tall's three worlds", however this is factually incorrect The paper with Eddie Gray, to which Inglis refers, speaks of 'three forms of mathematical concept' but *not* of 'three worlds of mathematics'.

I credit Eddie with all the inspiration for the notion of 'procept' and for maintaining my productivity in research in a wide range of areas over many years by his support and insight. Indeed, I am certain that, without his continuing wisdom and inspiration I would not have developed my ideas anywhere near getting close to the idea of 'three worlds'. However, the actual origins of the 'three worlds of mathematics' were in Watson (2002) and Watson *et al.* (2002)

The conceptual leap from three forms of concept to three worlds of mathematics may seem simple, but in practice it has proved to be both profound and daunting. It is one thing to have an insight into three different kinds of mathematical concept formed in different contexts; it is a much greater leap to claim that there are (at least) three distinct worlds of mathematics, each with a different mode of development and (in the formulation of Rodd (2000)) each with a different kind of warrant for mathematical truth.

Inglis's beautiful example of the imaginary sphere rightly challenges the published papers that referred to specializations in which such considerations are not discussed. However, within the wider theoretical framework, where his contribution is much appreciated, it is clearly an example arising out of the proceptual world of symbolic manipulation in which the algebra of spheres $x^2 + y^2 = r^2$ is applied to the case where $r^2 = -1$. In this context, the conceptually embodied meaning of spheres described algebraically no longer applies.

This is no different from the case of complex numbers, which, at their inception, had no conceptually embodied meaning. Even when complex numbers were visualized as points in the plane - as early as Wallis in his 1685 book on algebra, and diverse authors such as Wessel in 1797, Gauss in his 1799 doctoral thesis and Argand in 1806 re-invented complex numbers as points in the plane - it was still possible for De Morgan (1831) to state that the imaginary expression $\sqrt{(-a)}$ and the negative expression $-a$ indicated 'some inconsistency or absurdity [...] since $0 - a$ is as inconceivable as $\sqrt{(-a)}$'.

The imaginary sphere is part of the natural process of extending the manipulations of symbols that have meaning in the proceptual world to a situation where the corresponding link to the embodied world no longer holds. It is parallel to the idea of using the square root of a negative number as a manipulable symbol before it has a conceptual embodiment The fact that neither Inglis nor I can 'see' an embodiment, just as Cardan and Tartaglia could not 'see' a conceptual embodiment of $\sqrt{(-1)}$ is not a denial of the distinction between the conceptual-embodied world and the proceptual-symbolic world It is an affirmation that developments in the latter can operate independently of the former The proceptual world is *not* just an extension of conceptual embodiment, it has properties of its own which work (in a functionally embodied manner if you wish) in a way that need not have immediate counterparts in the embodied world

In his much-loved book on *The psychology of learning mathematics*, Skemp (1971) asserted that it is not possible to *define* higher-order concepts, what is necessary is for the individual to encounter examples of the concept to construct the higher-order meaning. Thus, in developing new higher-order concepts, he suggests it is necessary to start from a variety of good examples. I did this for the notion of different worlds of mathematics by looking at examples in very different contexts.

In building a new theory, it is usually the *total structure* that one is trying to create, including all the relevant concept imagery, not just a distilled definition. The formulation of a subtle cognitive theory cannot, therefore, be reduced to a short formal definition alone It needs to be established at a level where it applies to different contexts and links need to be made between ideas in a manner that is relevant to the overall theoretical perspective.

In my publications I have used the term 'embodiment' with a meaning that I believe is consistent with the colloquial notion of 'giving a body' to an abstract idea This includes all cases of conceptions in visuo-spatial terms, not only those that arise from perception of actual objects. As a result of Inglis's intervention, I have found that many individuals interpret my writing to refer only to perceptual embodiment. I have therefore moved to using a two-word definition to alert the reader to this fact and now use the name 'conceptual embodiment', at least in the initial stages, to alert the reader to my intended meaning

A word about words

Inglis closes his communication with another reference to the use of words, by referring to the 'frustrating habit' of Gray and Tall in using the words 'object' and 'concept' interchangeably. From our perspective we do not do that, so it might help to clarify. We use both words *in contexts* to express appropriate ideas. Mathematicians (such as Dales *et al* in1992) freely use the word 'object' for the things we talk about in formal mathematics, such as a 'group', 'matrix' or a 'topological space'. In colloquial language, however, we speak of counting processes and number *concepts*, not number *objects* For instance we use the term 'concept of number' and certainly not 'object of number'. We also speak of 'fraction concept' rather than 'fraction object' When the term 'concept' is used in this context, it therefore has the meaning that mathematicians consider to be an (abstract) object. However, it is not an object in the sense of a physical thing that we can perceive in the world.

To overcome this difficulty, I freely use the term 'concept' when I speak about numbers, fractions, algebraic expressions and so on, in a manner which fits with common useage but, at the same time, in a context where the symbol refers dually to process or concept (as a mental object). English is a language where we intentionally use the richness of diverse meanings to express rich ideas, which may be ambiguous if isolated as individual words but are intended to be made clear by the context.

In building a theory of different worlds of mathematics, therefore, I cannot begin by stating definitions and proving theorems. I have to begin with ideas that I test out by trying out formulations to see if they make sense to others and to test the ideas in several different contexts (so far, calculus, vectors, and proof) to see if they have a useful practical meaning In the examples I chose, it is clear from the response of many readers that the idea I expressed in terms of 'embodiment' has been made in a context that has overemphasized the relationship with perception of the physical world and at the expense of the longer-term conceptual embodiment of mental concepts as visuo-spatial concepts

I thank Inglis for his attention to the limitations in my initial examples and will continue to work at a broad theory of the growth of mathematical concepts that gives insight into the nature of mathematical growth [1].

Notes

[1] These ideas will be explored further in a book on mathematical growth that I am currently writing A draft of this book is available on the web, at davidtall com/mathematical-growth, in its partially complete state for private study and comment. The book is not yet ready for publication.

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