Two Approaches to Analyzing the Permutations of the 15 Puzzle

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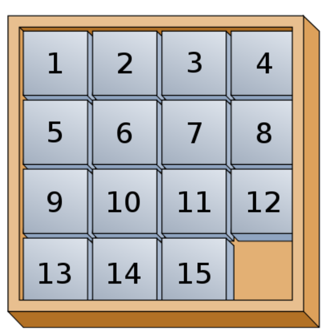
Abstract

The permutations of the 15 puzzle have been a point of focus since the 1880's when Sam Lloyd designed a spin-off of the puzzle that was impossible to solve. In this paper, we explore which permutations of the 15 puzzle are obtainable by utilizing properties of permutations and results from graph theory. We begin our investigation by using brute force to obtain elementary permutations of the puzzle followed by simple proofs to show that exactly half of the permutations of the 15 puzzle may be obtained. While this approach is sufficient to demonstrate the properties of the 15 puzzle, a more elegant proof using properties of simple graphs yields us results that may be extrapolated onto other similar-style puzzles that may be represented as simple graphs. The portion of this paper that centers around graph theory is an exposition of Richard Wilson's 1973 paper *Graph Puzzles, Homotopy, and the Alternating Group*.

1 Introduction

This paper will investigate which permutations of the 15 puzzle are obtainable. The rules of the puzzle are relatively simple: the original sequence of the numbers in the puzzle (pictured below) are scrambled, and the solver must restore the original configuration by strategically moving numbered squares into the empty space.



The 15 puzzle was invented by a postmaster named Noyes Chapman in Canastota, New York and was distributed by the Embossing Company around 1868. However, the puzzle rose to prominence in 1878 when a prominent American puzzle enthusiast named Sam Lloyd claimed that he had developed a new puzzle which was a modified version of the 15 puzzle [5]. However, the extent to which Lloyd contributed to the success of the puzzle in its early days is not certain. Lloyd's puzzle was titled the 14-15 puzzle and frustrated many curious minds who tried to solve it. The adaptation was identical to the original puzzle except that the numbers 14 and 15 were switched in order. As we will explore in this paper, restoring a puzzle configuration like this to the original sequence as shown in the figure above is impossible.

Lloyd was prepared to offer $\$1,000$ to anybody who could solve his tricky puzzle. The hype around the 15 puzzle was so significant starting in January 1880 that the inventor of the puzzle, Noyes Chapman, applied for a patent on the puzzle in March 1880. Nearly a century later, the inventor of the Rubik's Cube, Erno Rubik, claimed to have been inspired by the success of Lloyd's 14-15 puzzle to make another sensational puzzle [5].

Since Sam Lloyd's capitalization on the popularity of the puzzle, many other variations of the puzzle have arisen. Many of them form an image when completed or spell out a message.

Exactly half of all permutations of the 15 puzzle may be obtained. Only those configurations that require an even number of transpositions between the empty space and numbered tiles may be created. Aaron Archer proved in 1999 with a simple set of logical steps that only half of the configurations of the puzzle could ever be obtained by merely shuffling the tiles around. This short proof demonstrated that Sam Lloyd was a clever opportunist who took full advantage of the hype surrounding the puzzle by promising a juicy monetary prize to someone for accomplishing a task he knew to be impossible.

This paper considers two approaches to investigating the possible configurations of the puzzle: a graph theory approach and an abstract algebra approach. The approach from an abstract algebra perspective is specific to the 15 puzzle and involves the use of permutation groups and theorems. The graph theory approach will ask the reader to consider the puzzle as a 4×4 graph containing 16 vertices. One advantage to the graph theory approach is its generality relative to the abstract algebra approach.

2 Abstract Algebra Approach

2.1 Nine Key Permutations

The 15 puzzle's configurations are equivalent to the set of permutations of a set of 15 elements. That is, the set of permutations of the 15 puzzle can be thought of as the set S_{15}. Let us consider the puzzle's configuration where the open space is in the upper right corner, and the tile labeled "1" is in the square to its left. If numbers trail from "1" going from right to left in the first row, then left to right in the second, then right to left in the third, and left to right in the final row, then we have a sequence of consecutive numbers 1-15. See Figure 1 below for an illustration of this configuration.

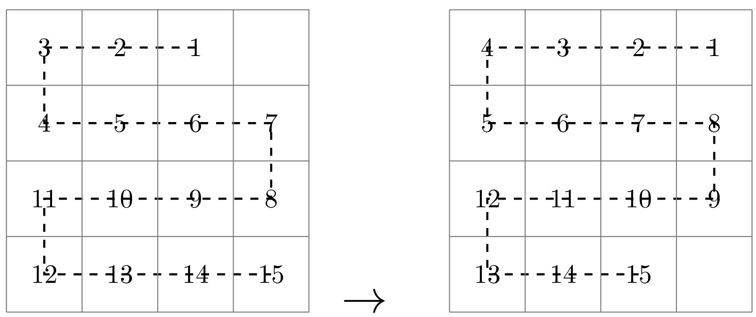


Figure 1: Typical 1-15 arrangement of the 15-puzzle. Notice that both puzzles feature the same permutation of the fifteen numbered squares despite the fact that the blank square has a different location in each configuration.

In this configuration of consecutive numbers, moving the empty square horizontally within a row does not change the permutation of the 15 elements. However, if the empty square is moved between rows (up and down), then it can change the permutation of the 15 elements. However, the permutation generated by this operation is even. Three examples of these permutations are given in Figures 2-4.

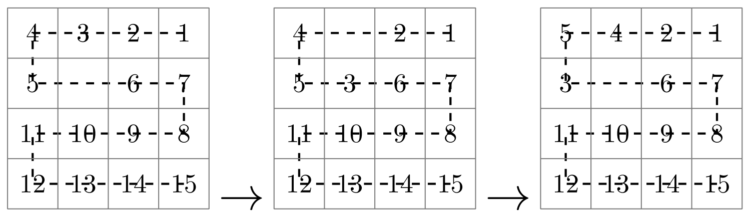


Figure 2:
The even permutation $(3, 5, 4)$ takes place when the empty space switches places with the number 3. 3 ends up in the old position of 5, 5 takes the place of 4, and 4 takes the place of 3.

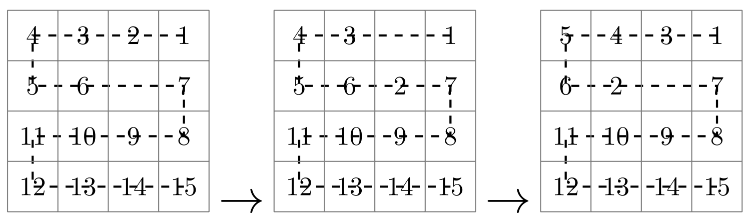


Figure 3:
The even permutation $(2, 6, 5, 4, 3)$ takes place when the empty space switches places with the number 2. 2 ends up in the old position of 6, 6 takes the place of 5, 5 takes the place of 4, 4 takes the place of 3, and 3 takes the place of 2.

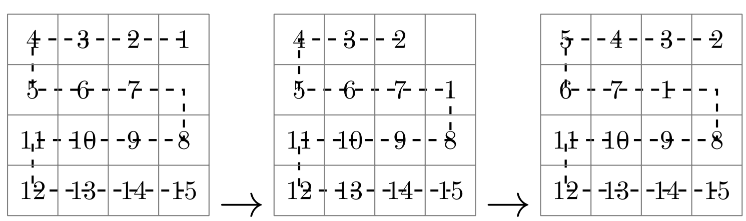


Figure 4:
The even permutation $(1, 7, 6, 5, 4, 3, 2)$ takes place when the empty space switches places with the number 1. The sequence of the permutation is determined using the same counting method from the previous two figures.

Since every movement of the empty square either preserves the current configuration or alters the configuration in a way characterized by an odd cycle, each movement of the empty square results in a permutation in which an even number of elements swap positions.

2.2 Analysis of Permutations

There are a total of nine permutations that can occur on the configuration of the puzzle in Figure 1. These permutations are as follows:

- $p_1 = (3, 5, 4)$
- $p_2 = (2, 6, 5, 4, 3)$
- $p_3 = (1, 7, 6, 5, 4, 3, 2)$
- $p_4 = (7, 9, 8)$
- $p_5 = (6, 10, 9, 8, 7)$
- $p_6 = (5, 11, 10, 9, 8, 7, 6)$
- $p_7 = (11, 13, 12)$
- $p_8 = (10, 14, 13, 12, 11)$
- $p_9 = (9, 15, 14, 13, 12, 11, 10)$

Note that each of these permutations are even since they contain an odd number of elements in their respective cycles. Since these are the nine permutations that arise from moving the empty space between rows of the puzzle, we see that the group generated from these nine permutations $< p_1, p_2, p_3, ..., p_8, p_9 >$ generate all permutations of the puzzle. According to Joseph Gallian's textbook *Contemporary Abstract Algebra 4th Edition*, the product of any number of even permutations is also even.

Notice that these nine permutations also form the set of consecutive 3-cycles of A_{15}:

- $(1, 2, 3) = p_3^2 p_1^2 p_3^{-2}$
- $(2, 3, 4) = p_3 p_1^2 p_3^{-1}$
- $(3, 4, 5) = p_1^2$
- $(4, 5, 6) = p_2^{-1} p_1^2 p_2$
- $(5, 6, 7) = p_6^2 p_4^2 p_6^{-2}$
- $(6, 7, 8) = p_6 p_1^2 p_6^{-1}$
- $(7, 8, 9) = p_4^2$
- $(8, 9, 10) = p_5^{-1} p_4^2 p_5$
- $(9, 10, 11) = p_9^2 p_7^2 p_9^{-2}$
- $(10, 11, 12) = p_9 p_7^2 p_9^{-1}$
- $(11, 12, 13) = p_7^2$
- $(12, 13, 14) = p_8^{-1} p_7^2 p_8$
- $(13, 14, 15) = p_9^{-1}(p_8^{-1} p_7^2 p_8) p_9$

So we see that we have every consecutive 3-cycle in our set of permutations of the puzzle. We must now prove that every 3-cycle in A_n can be obtained from products of the consecutive 3-cycles of A_n. That is, the set of three-cycles

$$(1, 2, 3), (2, 3, 4), ...(n-3, n-2, n-1), (n-2, n-1, n)$$

form all 3-cycles in A_n. Additionally, we must show that every permutation of A_n can be written as a product of 3-cycles.

Theorem 2.1. *Every permutation in A_n is a product of 3-cycles*

Proof. We will first show that every permutation of A_n can be written as a product of 3-cycles. First, suppose that $\alpha \in A_n$. We note that we can write every permutation as a product of 2-cycles (Gallian 103). Let α be written as a product of 2-cycles $(x_1, y_1)(x_2, y_2), ...(x_k, y_k)$. k must be an even number since it is expressing an even permutation α as a product of odd cycles. Therefore, it is possible to group the transpositions into pairs and consider any potential pairs of 2-cycles. There are three possible cases that arise in each pair.

Case 1: The two 2-cycles in the pair share the same elements- If this is the case, then the pair of 2-cycles may be removed from the product since $(ab) * (ab) = e$.

Case 2: The two 2-cycles in the pair share one element, but each 2-cycle has a distinct element- In this instance, the pair of 2-cycle can be written as a 3-cycle like so: $(ab) * (bc) = (acb)$.

Case 3: The two 2-cycles in the pair do not share a common element- If the two 2-cycles are of the form $(ab) * (cd) = (bdc)(acb)$.

Thus, every permutation $\alpha \in A_n$ can be written as a product of 3-cycles. Now, let us turn our attention to proving that the set of consecutive 3-cycles of A_n can generate all others. ☐

Theorem 2.2. *The set of consecutive 3-cycles of A_n can form all other 3-cycles of A_n*

Proof. We will prove that the set of consecutive 3-cycles of A_n generates all other 3-cycles of A_n by induction. Let us consider the base case A_3. We begin with A_3 because this is the smallest group of even permutations with a 3-cycle. Looking at the 3-cycle (1 2 3), we observe that we may obtain the other 3-cycle in A_3: $(1\ 2\ 3)^2$=(1 3 2). We have obtained all 3-cycles in A_3.

Let us prove another base case, A_4, for good measure. The consecutive 3-cycles in A_4 are (1 2 3) and (2 3 4). We may obtain all other 3-cycles in A_4 via the following operations:

- $(1, 2, 3) = (1, 2, 3)$
- $(1, 3, 2) = (1, 2, 3)^2$
- $(1, 2, 4) = (1, 2, 3)(2, 3, 4)^2$
- $(1, 4, 2) = (2, 3, 4)(1, 2, 3)^2$
- $(1, 4, 3) = (2, 3, 4)^2(1, 2, 3)$
- $(1, 3, 4) = (1, 2, 3)^2(2, 3, 4)$
- $(2, 3, 4) = (2, 3, 4)$
- $(2, 4, 3) = (2, 3, 4)^2$

Let us now assume that the theorem holds true for some positive integer k where $k \geq 3$. Let us then consider the $k+1^{st}$ case: We have the set of consecutive 3-cycles

$$(1,2,3),(2,3,4),...,(k-3,k-2,k-1),(k-2,k-1,k),(k-1,k,k+1)$$

Our mission is to show that we may create any 3-cycle with $k+1$ in it from this list. Note that our inductive hypothesis allows us to assume that the 3-cycles, excluding $(k-2,k-1,k)$ generate all 3-cycles in A_k. Let us consider three cases to prove that any 3-cycle containing $k+1$ is obtainable.

Case 1: Generate the 3-cycle with the form (a,k+1,k):

$$(a,k,k-1)*(k-1,k,k+1) = (a,k,k+1)$$

Where $a < k-1$. Note that our obtained 3-cycle may be squared to give us $(a,k+1,k)$.

Case 2: Generate the 3-cycle with the form $(a,k+1,k-1)$:

$$(a,k-1,k)*(k,k-1,k+1) = (a,k-1,k+1)$$

Where $a < k-1$. note that our obtained 3-cycle may be squared to give us (a,k+1,k-1).

Case 3: Generate the 3-cycle with the form (b,a,k+1):

$$(b,a,k)*(a,k+1,k) = (a,k+1,b)$$

Where $a,b < k-1$. We borrowed our 3-cycle $(a,k+1,k)$ from the result of case 1. Also observe that our obtained 3-cycle may be squared to give us $(a,b,k+1)$.

We have now proved that for any A_n where $n \geq 3$, the set of consecutive 3-cycles generates every 3-cycle in □

The results above indicate that the set of permutations obtained by moving the blank space between rows in the puzzle yield all possible permutations of A_{15}. Thus, all even permutations of the 15 puzzle are possible to obtain. However, none of the odd permutations may be obtained.

3 Graph Theory

Another way of thinking about the 15 puzzle is through graph theory. The puzzle is essentially a 4×4 grid with 16 vertices. The paper "Graph Puzzles, Homotopy, and the Alternating Group", authored by Richard M. Wilson provides a series of proofs to theorems which delve into the nature of the 15 puzzle. The paper focuses on methods of labeling a graph with 15 elements and a blank vertex. Sliding elements into the blank vertex yields a blank vertex in the formerly occupied vertex.

Definition 3.1 (Labeling)**.** A labeling on G is a bijective mapping $f : V(G) \to \{1, 2, ..., n, \emptyset\}$.

The vertex x with $f(x) = \emptyset$ is said to be unoccupied in f. That is, x is the empty vertex. [4]

Definition 3.2 (Adjacent Labeling)**.** Labelings f, g on G are adjacent if and only if g can be obtained from f by sliding a label along an edge of G onto the unoccupied vertex.

This relation of adjacency of labelings is symmetric and irreflexive and defines a simple graph denoted puz(G). The vertices of puz(G) are labelings of G, and two vertices of puz(G) are connected by an edge if and only if they are adjacent. Each component of puz(G) is composed of labelings that may be obtained from a series of "slides" which swap the empty vertex with an adjacent vertex's label. We now define a couple additional terms before proceeding with the proof.

Definition 3.3 (Bipartite Graph)**.** A graph in which the vertices may be partitioned into two parts denoted A and B such that no two vertices within A (B) are adjacent.

Definition 3.4 (Non-Separable Graph)**.** Any graph that may not be disconnected by removing one vertex.

Theorem 3.1. *Let G be a finite simple non-separable graph other than a polygon or a theta graph. Then puz(G) is connected unless G is bipartite, in which case puz(G) has exactly two components. In this latter case, labelings f,g on G having unoccupied vertices at even (or respectively, odd) distance in G are in the same component of puz(G) if and only if fg^{-1} is an even (or respectively, odd) permutation of $V(G)$. [4]*

The proof of this theorem will support our previous finding that exactly half of the permutations of the 15 puzzle may be obtained starting from any given arrangement. We first outline key definitions and put forward two propositions.

Let us define a path p in a graph G as a sequence

$$p = (x_0, x_1, x_2, ..., x_n)$$

of vertices of G where x_{i-1} and x_i are adjacent in G, $i = 1, 2..., n$. We call p a path from the initial vertex, x_0, to the final vertex x_n. A path p is said to be simple when $x_0, x_1, x_2, ...x_n$ are distinct with the possible exception that $x_0 = x_n$ in which case p is a simple closed path. Also define $\rho = (x_n, ..., x_2, x_1, x_0)$ to be the reverse of p.

For each $p = (x_0, x_1, x_2, ..., x_n)$ in G, let us define a permutation σ_p as the product of 2-cycles shown below:

$$\sigma_p = (x_{n-1}, x_n), ..., (x_2, x_3,)(x_1, x_2), (x_0, x_1)$$

The following two propositions are evident from these definitions.

Proposition 1: Labelings f, g on a graph G are in the same component of $puz(G)$ iff $f = g\sigma_p$ for some path p in G from $f^{-1}(\emptyset)$ to $g^{-1}(\emptyset)$.

Let us define $\Gamma(x, y) = \Gamma_G(x, y)$ to be the set of all permutations σ_p of $V(G)$ where p is a path from x to y in G. We will abbreviate $\Gamma(x, y) = \Gamma_G(x, y)$ with the expression $\Gamma(x) = \Gamma_G(x)$ if $x = y$.

Note that paths can be written transitively. For instance, if p is a path from x to y and q a path from y to z, then one of the paths from x to z is the product qp. In other words, $\sigma_p\sigma_q = \sigma_{qp}$. We are now ready to consider the second proposition.

Proposition 2: For each vertex x of G, $\Gamma(x)$ is a group of permutations of $V(G)$, each fixing x. If p is a path from x to y in G, then $\Gamma(x, y) = \sigma_p\Gamma(x) = \Gamma(y)\sigma_\rho$ and $\Gamma(y) = \sigma_p\Gamma(x)\sigma_p^{-1}$.

These propositions, conjunction with Theorem 3.2 (below) and an assortment of lemmas prove Theorem 3.1.

Theorem 3.2. *Let G be a finite simple graph other than a polygon or the graph θ_0. Then, for any vertex x of G,*

$$\Gamma(x) = \mathit{sym}(V(G) - \{x\}),$$

unless G is bipartite, in which case

$$\Gamma(x) = \mathit{alt}(V(G) - \{x\}).$$

Note that $\mathrm{sym}(X)$ denotes the symmetric group of labelings on the set X while $\mathrm{alt}(X)$ denotes the alternating group of labelings on the set X where X is the set of vertices of G. Because of our propositions, it will suffice to show that

$$\mathrm{alt}(V(G) - \{x\}) \subset \Gamma(x).$$

Let us now prove a theorem which will help us prove Theorem 3.2 by induction.

Definition 3.5 (Arc)**.** An arc is a finite tree with exactly two monovalent vertices–namely, its ends.

Theorem 3.3 (The Handle Theorem)**.** *Let G be a non-separable, simple graph with at least three vertices and suppose that K is a non-separable proper subgraph of G with non-empty edge set. Then we can write $G = H \cup A$, where H is a non-separable subgraph of G containing K, A an arc-subgraph of G, and $H \cap A$ consists only of the ends of A.*

Proof. This proof will require the consideration of two cases:

Case 1: $|V(H)| = |V(G)|$
In this instance, we notice that the subgraph H contains every vertex that G contains. Since H is a proper subgraph of G and contains all vertices of G, it may only omit one edge of G at most. If H omitted more than one edge of G, then we would be able to find a larger subgraph of G and brand the new subgraph H.

Case 2: $|V(H)| < |V(G)|$
In this instance, we notice that the subgraph H does not contain every vertex that G contains. Suppose that a vertex v is in $V(G)$ but not $V(H)$. Let u be a vertex of H. Since G is 2-connected, there is a cycle C containing v and u. Following this cycle from v to u, Let w be the first vertex in H. Continuing on the cycle from u to v, let x be the last vertex in H. If $x \neq w$, let A be the path $(x, v_1, v_2, \ldots, v_k, v = v_{k+1}, v_{k+2}, \ldots, v_m, w)$, that is, the portion of the cycle between x and w containing no vertices of H except x and w. Since H together with A is 2-connected, it is G, as desired [2].

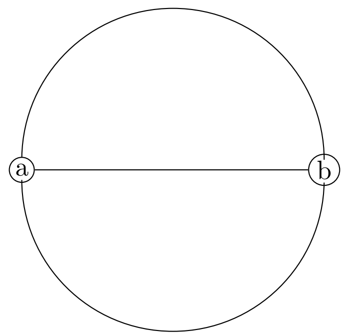
If $x = w$ then $x = w = u$. Let y be a vertex of H other than u. Since G is 2-connected, there is a path P from v to y that does not include u. Let v_j be the last vertex on P that is in $\{v_1, \ldots, v, \ldots, v_m\}$; without loss of generality, suppose $j \geq k+1$. Let z be the first vertex on P after v_j that is in H. Then let A be the path $(u, v_1, \ldots, v = v_{k+1}, \ldots, v_j, \ldots, z)$, where from v_j to z we follow path P. Now $H \cup A$ is a 2-connected subgraph of G, but it is not G, as it does not contain the edge $\{u, v_m\}$, contradicting the maximality of H. Thus $x \neq w$ [2]. $\square$

The Handle Theorem provides the basis of induction that we perform on simple, non-separable graphs. We make the case that any non-separable graph is either bipartite or not bipartite. Thus, every simple, non-separable graph G is such that all labelings may be obtained from an initial labeling or only half may be obtained. The only two exceptions to this are the polygon graphs and the graph θ_0 as shown below. However, the purpose of the following section is to demonstrate that when pulling handles off of non-separable, simple graphs, we can always remove a handle such that θ_0 is not obtained. Before we launch into this induction, let us define a measure of complexity for graphs.

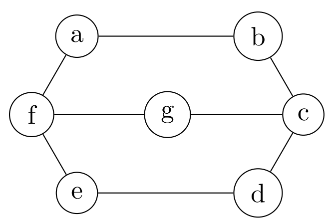
Definition 3.6 (Betti number)**.** The Betti number is a measure of the complexity of graphs. A given graph G has a Betti number denoted β and is defined as follows:

$$\beta(G) = |E(G)| - |V(G)| + 1.$$

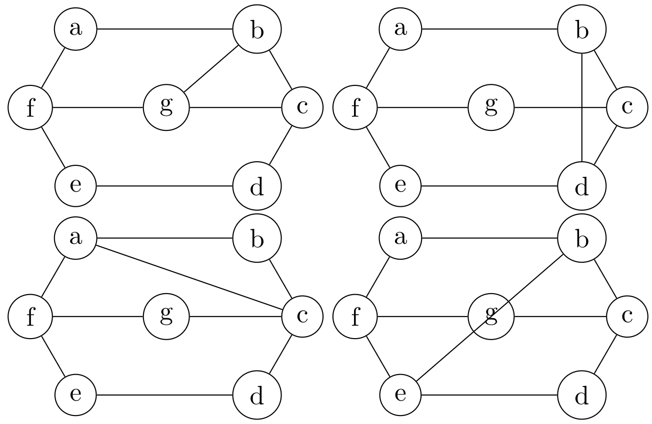
If we add a handle (an arc) to a polygon, we obtain a group of graphs called the $\theta-$graphs which are non-separable graphs with two vertices of degree three which have exactly three paths connecting them. θ-graphs also have a Betti number of 2. The most simple of these graphs, we will call it U, is pictured below.



Every θ-graph is a subdivision of U. In other words, every θ-graph may be obtained by dividing the three arcs of U into multiple edges by inserting vertices on each arc. We will first note that Wilson proves that Theorem 3.2 is true for θ-graphs. Thus, Theorem 3.2 is true for graphs with a Betti number of 2. We now suppose that Theorem 3.2 is true for some graph G with $\beta(G) \geq 3$. Let us write G as $H \cup A$ where H is a non-separable subraph of G with $\beta(H) = \beta(G) - 1$ and A is an arc with only its end vertices in H. Below, we demonstrate that we may choose H to be a graph other than θ_0 (pictured below).

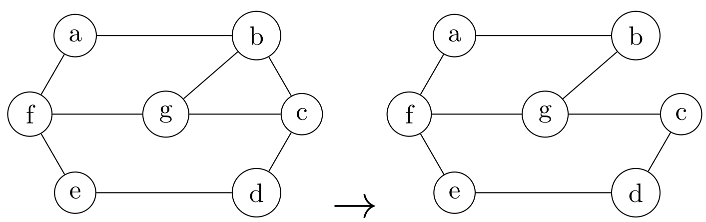


Note that we are only at risk of picking θ_0 to be H if G has Betti number 3 and is of the form of one of eight graphs. Four of these graphs pictured below. We show further below that we may avoid picking θ_0 to be H by strategically choosing which arc we designate to be A.



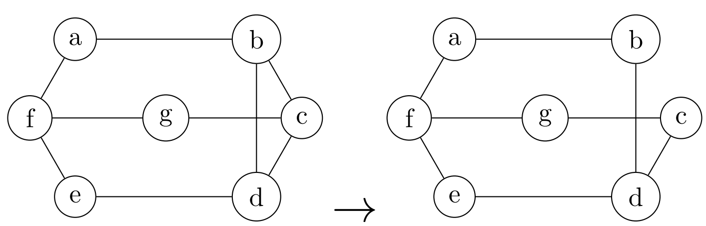
Furthermore, each of these graphs can have one arc removed to yield a $\theta - graph$ that is not θ_0. We will address each of the four graphs one case at a time.

Case 1:



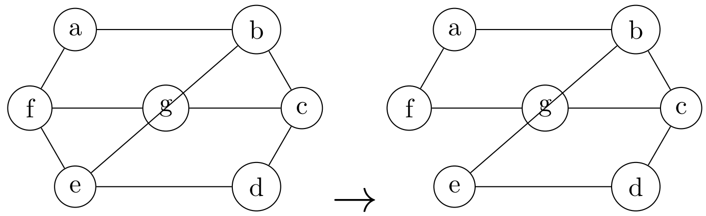
The resulting graph is a $\theta - graph$ that is not θ_0. This graph is designated as H while the arc connecting vertices b and g is designated as the arc A.

Case 2:



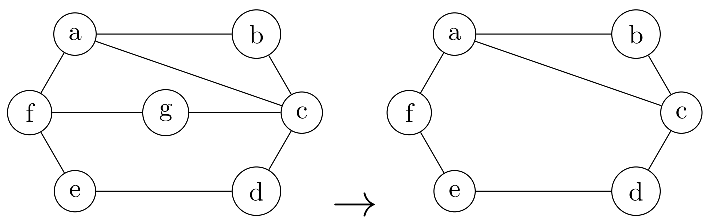
The resulting graph is a $\theta - graph$ that is not θ_0. This graph is designated as H while the arc connecting vertices b and c is designated as the arc A.

Case 3:



The resulting graph is a $\theta - graph$ that is not θ_0. This graph is designated as H while the arc connecting vertices e and f is designated as the arc A.

Case 4:



The resulting graph is a $\theta - graph$ that is not θ_0. This graph is designated as H while the arc connecting vertices c and f is designated as the arc A.

The remaining four cases are similar to the four shown, and the process of removing an arc to yield a θ-graph that is not θ_0 is also similar to the processes shown above.

Now that we have established that H need not be θ_0, we may proceed with induction assuming that Theorem 2 holds for H. With Theorem 2 established for the graph H, we can write alt$(V(G) - \{x\}) \subset \Gamma_H(x)$ for each $x \in V(H)$. There is a result in graph theory analogous to our earlier proof that the group of three cycles of a set of permutations form the whole group A_n. Wilson gives a short proof of this in his paper. We now borrow one more theorem from Wilson.

Theorem 3.4 (Theorem 4)**.** *Let Γ be a transitive permutation group on X and suppose that Γ contains a 3-cycle. If Γ is primitive (in particular, if Γ is double transitive), then alt$(X) \subset \Gamma$.*

Since $\Gamma_G(x)$ contains $\Gamma_H(x)$ which contains 3-cycles, we can invoke the result of Theorem 4 and merely show that $\Gamma_G(x)$ is doubly transitive on $V(G) - \{x\}$. Wilson demonstrates that this fact is true, and thus we have proven Theorem 2 for any graph G for which $\beta(G) \geq 2$ excluding θ_0. Note that graphs with a

Betti number of 1 are polygon graphs which are excluded in the hypotheses for Theorems 3.1 and 3.2. Theorem 3.1 follows closely from Theorem 3.2.

Because the 15-puzzle may be represented as a collection of 16 vertices arranged in a 4×4 grid which is a bipartite graph, only half of its vertex lableings may be obtained given a starting configuration. This furthers our findings from the abstract algebra section of this paper. Because resolving the 14-15 puzzle would require a single transposition (which means that this would be an odd permutation of the tiles), the original configuration of the 15 puzzle may not be obtained from the original configuration of the 14-15 puzzle.

References

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The 15-puzzle has been an object of great mathemat-ical interest since its invention in the 1860s. The puzzle has 16 square slots on a square board. The rst 15 slots have square pieces; the 16th slot is empty.Â A permutation of a set is a bijection from onto itself. If the set we are permuting is = {1, 2, . }, it is often convenient to represent a permutation as follows: (= 1 (1). 2 (2).Â Cycles which consist of only two elements, such as the cycle (3 5) in , are referred to as 2-cycles, or more commonly, as transpositions. The transposition (3 5) can also be written as (5 3), as both have the eect of swapping the elements 3 and 5. Subsequently, any transposition is its own inverse. Theorem 1.1. The 15-puzzle is a well-known game which has a long history stretching back in the 1870's. The goal of the game is to arrange a shuffled set of 15 numbered tiles in ascending order, by sliding tiles into the one vacant space on a 4x4 grid.Â Sliding puzzles on graphs are generalizations of the Fifteen Puzzle. Wilson has shown that the sliding puzzle on a 2-connected graph always generates all even permutations of the tiles on the vertices of the graph, unless the graph is isomorphic to a cycle or the graph Î¸0 [R.M. Wilson, Graph puzzles, homotopy, and the alternating group, J. Combin. Theory Ser. B 16 (1974) 86â€"96].Â In addition to the novel multi-robot metrics, a central contribution of this work are tools to analyze and predict the effectiveness of metrics in the MRMP context.