

THE “COEXISTENCE PROBLEM” FOR CONSERVATIVE  
DYNAMICAL SYSTEMS: A REVIEW

BY

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Któż zbadał puszczy litewskich przepastne krainy,  
Aż do samego środka, do jądra gęstwiny?  
Rybak ledwie u brzegów nawiedza dno morza;  
Myśliwiec krąży koł o puszczy litewskich łąka,  
Zna je ledwie po wierzchu, ich postać, ich lice,  
Lecz obce mu ich wewnętrzne serca tajemnice:  
Wieść tylko albo bajka wie, co się w nich dzieje.

A. Mickiewicz, *Pan Tadeusz*, księga IV <sup>(1)</sup>

**1. Introduction.** This is an informal and very concise personal account about apparently simple conservative examples studied numerically in view of the “coexistence problem”. My aim is to stress the fundamental character of this problem and to indicate some appropriate examples together with bibliographical references. I do not attempt to make this review exhaustive. Taking into account the tremendous development of research in this area, this is simply impossible, especially in so short a note. Nevertheless, the reported bibliography together with the references that one can find in the cited works cover a large part of publications on the subject.

This paper is addressed mainly to people interested in nonlinear dynamics (from a theoretical as well as from a numerical point of view).

Throughout the paper, conservative means preserving a smooth measure. When the phase space is compact, this measure needs to be finite. In all reported examples this measure is always either Lebesgue measure or Liouville measure on a constant energy surface. When speaking about

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<sup>(1)</sup>“Who has explored the deep abysses of the Lithuanian forests up to the very centre, the kernel of the thicket? A fisherman is scarcely acquainted with the bottom of the sea close to the shore; a huntsman skirts around the bed of the Lithuanian forests; he knows them barely on the surface, their form and face, but the inner secrets of their heart are a mystery to him; only rumour or fable knows what goes on within them.” A. Mickiewicz, *Pan Tadeusz*, book IV (translated from Polish by George Rapall Noyes, J. M. Dent&Sons Ltd., London and Toronto 1917).

metric entropy, we always mean the entropy with respect to such a measure.

General aspects of numerical study of chaos in conservative dynamical systems are described from the physicists' point of view in the book [58]. In that book one can find many concrete examples with numerically observed coexistence as well as a good bibliography up to 1983.

Let us indicate also four more recent books [70], [81], [82] and [10], also written from the physicists' point of view ([70] is an extended English translation of [82]).

The background required for reading the present paper is very limited. Even a very fragmentary knowledge of any of these books is more than sufficient to understand the major part of this review.

For me, a conservative dynamical system presents *coexistence* if the phase space is a union of two subsets, both of positive Lebesgue measure, on one of them all Lyapunov exponents are zero and on the other the maximal Lyapunov exponents are strictly positive. Moreover, to avoid "simple coexistence" we need that both sets be dense in the phase space. In this last case we will speak about "true coexistence", shortly TC. The set with strictly positive Lyapunov exponents is also called a chaotic region and the behaviour of the system on it is called chaotic.

The coexistence phenomenon was discovered numerically by M. Hénon and C. Heiles in 1963 and published in their epoch-making paper [40]. From that time, the incredible amount of numerical computations suggest that typically TC takes place in conservative dynamical systems. The *coexistence problem* consists in proving this. Up to now, *one does not know any example where this is proved*. To my knowledge this kind of problem was first stated by Ya. G. Sinai at the latest in 1969 (see Sec. 2.5 of [17] and Sec. 4 of [19]).

My terminology about "simple coexistence" is in fact misleading because it is not easy to find an example with this phenomenon. The first explicit smooth example of this kind was given in 1982 by F. Przytycki ([65]). Other, piecewise linear, examples with coexistence can be found in [77].

We would like to stress the fact that, except for the above quoted examples and similar ones, we do not know any other concrete example where coexistence is proved <sup>(2)</sup>. The lack of methods allowing to prove coexistence is the main mathematical problem in the area.

This problem is intimately related to the problem of recognizing systems of linear differential equations having a positive maximal Lyapunov exponent. Even in the two-dimensional case this problem is largely open (cf. [31], see also [42]).

Thus we have here a truly paradoxical situation, as the famous Pesin theory ([64], [47], [66]) gives a quite satisfactory description of the behaviour

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<sup>(2)</sup> See Note added in proof.

of a system restricted to its chaotic region, at least in low dimensions. As far as the set of positive Lebesgue measure with vanishing Lyapunov exponents is concerned, up to now its existence in smooth systems of the kind discussed in this paper has always been obtained using the Kolmogorov–Arnold–Moser (KAM) theory.

Let us note that by the Pesin entropy formula ([64], [47]) coexistence implies that the metric entropy of the system is strictly positive. Except for some piecewise linear examples and for examples from [65], also this last property is not proved for any of the concrete examples described in this survey.

In what follows, I will not consider the problem of proving the existence of homoclinic orbits and of the associated shifts (cf. [62], [76]), nor the problem of nonexistence of analytic first integrals (cf. [49]), despite their evident relation to the coexistence problem. I will not consider any more the problem of coexistence in the vicinity of a generic elliptic fixed point (cf. [16], [62], [83]), or for  $C^\infty$  generic perturbations of the standard twist map (cf. [34], [56]), although these seem to be the major problems in this area. I apologize also for the absence of the Newtonian many-body problem (cf. [1]) in this note; that topic deserves a separate review.

I will consider different examples in the following order: standard maps, piecewise linear mappings, piecewise smooth mappings and billiards, smooth mappings, and finally flows, especially Hamiltonian ones. As far as the mappings are concerned, I will consider only the two-dimensional case because, at least nowadays, the understanding of the coexistence phenomenon for them seems to be the heart of the problem.

Finally, let us note that a large part of the contemporary studies of nonlinear systems is devoted to dissipative systems and their (strange) attractors (cf. excellent collection of reprints [24], see also [44]). If we consider such an attractor as the space itself on which the system acts, forgetting the ambient space, we are, roughly speaking, in a virtually conservative situation. In this framework it is natural to ask about coexistence with respect to the natural measures (invariant or not) on the attractor.

It is a pleasure for me to thank G. Benettin (University of Padova), L. Galgani and A. Giorgilli (University of Milan), who introduced me to the subject, as well as to A. Katok (Pennsylvania State University), who taught me the Pesin theory. I also thank all of them for many years of collaboration. I thank D. Ornstein (Stanford University) and J.-P. Thouvenot (University of Paris 6), who inspired me with the idea of writing this note. I thank R. Douady (École Polytechnique, Palaiseau), L. M. Eliasson (Stockholm University), P. Biler (Wrocław University) and T. Nadzieja (Wrocław University) for the critical reading of the manuscript. I thank also G. Con-

topoulos (University of Athens) for his help in bibliographical queries.

**2. Standard maps.** It seems that the so-called “standard maps” and their modifications are the most studied, numerically as well as theoretically, two-dimensional conservative mappings. These maps were introduced for the first time by the physicists B. V. Chirikov and J. B. Taylor. A general review concerning these mappings can be found in [59]; besides, the whole volume where that paper is published contains many other examples of systems of physical interest with numerically observed TC.

Perhaps the simplest example of such a mapping is the mapping of the two-dimensional torus  $T^2$  given by the formula

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \varepsilon \sin 2\pi(x + y) \\ x + y \end{pmatrix}, \quad (x, y) \bmod 1, \quad \varepsilon \in \mathbb{R}.$$

When  $\varepsilon = 0$ , one obtains a linear twist map. When  $\varepsilon \neq 0$  one observes numerically TC. The same is true for the standard map in the proper sense, i.e. for the map ([17], [18], [58], [75])

$$(1) \quad T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \varepsilon \sin 2\pi y \\ x + y + \varepsilon \sin 2\pi y \end{pmatrix}, \quad (x, y) \bmod 1,$$

where  $\varepsilon \in \mathbb{R}$ . More generally, one can consider the mapping

$$(2) \quad T_f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + f(y) \\ x + y + f(y) \end{pmatrix}, \quad (x, y) \bmod 1,$$

where  $f$  is a smooth periodic function of period one.

There are also various modifications of maps of this kind, for example the mapping

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \varepsilon \sum_{k=1}^m \sin \pi k(x + y) \\ x + y \end{pmatrix}, \quad (x, y) \bmod 1, \quad \varepsilon \in \mathbb{R}, \quad m \geq 1$$

studied in [84], where the existence of a homoclinic point for  $\varepsilon > 18$  is proved. Among the rigorous results concerning the standard maps which are somewhat related to the problem of coexistence let us mention the papers [34], [56] and [45], [46]. The first two of them contain examples of standard maps with invariant Cantor set of the Aubry–Mather type with nonvanishing Lyapunov exponents (found also by M. Herman (unpublished)). Unfortunately, as proved in [45], the Lebesgue measure of the union of all such sets is always zero. See also [51]–[55] and [73], where the non-integrability versus integrability problem for the standard map is investigated. Another important study of the standard map related to the ideas of the KAM theory can be found in [61].

**3. Piecewise linear mappings.** Let us replace in (1) the sine (or  $f$  in (2)) by a continuous piecewise linear function with zero mean and period

one. Maps of this kind were intensively studied by M. Wojtkowski ([77], [78], [22]). Using his criterion of positivity of the maximal Lyapunov exponent ([77]–[80], [22]), he gave proofs of the coexistence for many particular cases in this setting. But he did not succeed in proving TC, which surely takes place in many examples of this kind. See also [25], where the coexistence is proved for the area preserving map

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - y + |x| \\ y \end{pmatrix}$$

of the plane. A study of such examples from the point of view of the existence of invariant curves as well as of counterparts of the Aubry–Mather theory is given in [14]. Let us remark that in many other smooth examples it can be interesting to consider the piecewise linear approximation similar to the one above (cf. [57]).

**4. Piecewise smooth mappings and billiards.** First let us consider the continuous case. The typical example of a continuous and piecewise smooth mapping is a natural Poincaré section map for a strictly convex plane billiard whose boundary is of class  $C^1$  and is a union of four arcs of circles. Perhaps this is a simplest genuinely non-linear conservative example in which one numerically observes TC. These examples were studied in [9], [41] and [36]. In [36] one can find very interesting pictures of incredible complexity. See also [27] and [50] for theoretical investigations of strictly convex billiards.

As far as I know, there are no examples of strictly convex  $C^\infty$  billiards with positive metric entropy. The problem of constructing such billiards can be considered as a variation of the already formulated problem of the coexistence for  $C^\infty$  generic perturbations of the standard twist map.

For example, the strictly convex billiard with even real analytic boundary defined by the formulas

$$\begin{aligned} x(t) &= \cos t + \lambda \cos 2t, \\ y(t) &= \sin t + \lambda \sin 2t, \end{aligned} \quad 0 < \lambda \leq 1/4,$$

seems to be a good candidate for having positive metric entropy. See [68] and [37] for details.

Let us note also another outstanding open problem concerning convex billiards. It is well known that the billiards in the ellipse are integrable. More precisely, the envelopes of the reflecting rays inside an ellipse coincide with the confocal ellipses. Is the ellipse the unique smooth closed convex curve with the property that the envelopes of the reflecting rays inside it are always closed convex curves? See [21] for this and other related problems.

As far as the coexistence problem is concerned, the continuity assump-

tion does not seem to be essential (cf. [47]). When one considers the natural Poincaré section for not strictly convex billiards, the mapping obtained is never continuous. A beautiful example of this kind with numerically observed TC is studied in [71]. Other interesting examples of piecewise smooth maps of the plane which preserve Lebesgue measure can be found in [32]. See also [43] for their theoretical study.

**5. Smooth mappings.** Besides the standard maps already discussed and their modifications, another interesting class of maps preserving Lebesgue measure can be defined as follows. Just as the mapping (2), one can consider the diffeomorphisms of two-dimensional torus introduced for the first time by D. V. Anosov in Sec. 25 of [2] and defined by the formula

$$R_f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + y + f(2x + y) \end{pmatrix}, \quad (x, y) \bmod 1,$$

where  $f$  is a smooth periodic function of period 1.  $R_f$  is a perturbation of the Anosov map  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  and one can check that if  $\|f'\|_\infty \leq 1/5$  then  $R_f$  is still an Anosov map. Hence there is no coexistence, because for an Anosov map, the maximal Lyapunov exponent is strictly positive everywhere. But what happens for large perturbations? The numerical computations indicate, for example, that if  $f(t) = \varepsilon \sin 2\pi t$  with  $|\varepsilon|$  big enough, we are in the presence of coexistence. More precisely, for  $-5/2\pi < \varepsilon < -1/2\pi$ ,  $(0, 0)$  is an elliptic fixed point. As follows from the KAM theory, generically this point is surrounded by invariant curves which cover a set of positive Lebesgue measure. It is precisely among the maps of type  $R_f$  that F. Przytycki found his example of coexistence ([65]) already mentioned in Section 1. It seems to me that these maps have never been studied in the literature except [2] and [65].

An interesting related question is the following. Is it possible to destroy completely the chaos by sufficiently perturbing the mapping  $R_0$ ? More precisely, is it possible to find a Lebesgue measure preserving diffeomorphism of the two-dimensional torus which is isotopic to  $R_0$  and of zero metric entropy? By the Pesin entropy formula ([64], [47]), the metric entropy vanishes if and only if the Lyapunov exponents are zero almost everywhere.

Except for standard maps and other maps related to them, as well as for maps like  $R_f$  defined above, it is difficult to indicate another so interesting class of conservative mappings with numerically observed TC. There are plenty of such conservative mappings with numerically observed TC, but there is no apparent reason to consider one more interesting than another.

The following example is of somewhat different nature. M. Hénon ([39]) studied numerically the quadratic mapping

$$T_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \alpha - (y - x^2) \sin \alpha \\ x \sin \alpha + (y - x^2) \cos \alpha \end{pmatrix}.$$

$T_\alpha$  is a diffeomorphism of  $\mathbb{R}^2$  and preserves Lebesgue measure. The origin is an elliptic fixed point of  $T_\alpha$ , and in general this fixed point is surrounded by invariant curves (KAM). Hénon's computations suggest that at least in the vicinity of the origin we observe the presence of TC. Another similar mapping of the plane

$$M_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x + y^3) \cos \alpha - y \sin \alpha \\ (x + y^3) \sin \alpha + y \cos \alpha \end{pmatrix}$$

was introduced by R. Cushman ([23]). In a small neighbourhood of the origin, we return to the already formulated problem of coexistence in the vicinity of the general elliptic fixed point. As these examples are defined on  $\mathbb{R}^2$  and some points have unbounded orbits, the global picture could be even much more complicated than in the preceding examples.

**6. Flows.** As far as flows are concerned, I shall describe now several apparently simple systems of differential equations in small dimensions with numerically observed TC. I shall be mainly interested in Hamiltonian flows on compact energy surfaces.

a) *Hénon–Heiles system* ([40]). As already stated this is historically the first studied conservative (Hamiltonian) system with numerically observed TC. It corresponds to the Hamiltonian

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3} q_2^3.$$

For each energy level  $E$ ,  $0 < E < 1/6$ , the surface of constant energy  $E$  contains the unique compact component  $M_E$ , and, when speaking of a Hénon–Heiles system, one means the restriction of this system to  $M_E$ . One observes numerically TC on  $M_E$ . For details see [58], [20], [69] and [8].

b) *Störmer problem*. This problem arising from the study of electrically charged particles in a magnetic field is discussed in [13], where one can find also other references. The lack of place prevents me from a more thorough discussion of this problem.

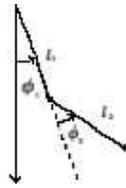
c) *Unequal-mass Toda lattice*. The corresponding Hamiltonian is

$$H(p, q) = \frac{1}{2}(p_1^2/m_1 + p_2^2/m_2) + e^{-q_1} + e^{q_1 - q_2} + e^{q_2} - 3,$$

where  $m_1 > 0$ ,  $m_2 > 0$ . When  $m_1 = m_2$  this is a two-particle Toda lattice, which is completely integrable ([74]). When  $m_1 \neq m_2$  and  $E$  is large enough, G. Casati and J. Ford ([15]) observed numerically TC. This is particularly interesting because O. I. Bogoyavlensky ([11]) gave serious theoretical arguments which confirm that for sufficiently large energy  $E$ , the chaotic behaviour takes place in this system. His arguments are described in much detail in his book [12]. That very important book contains also

a description of a great number of very interesting dynamical systems and theoretical arguments for the existence of chaos in them.

d) *Double pendulum* ([67]). Let us consider in a vertical plane the double pendulum as in the figure.



gravitational force

After some scaling the Hamiltonian of the double pendulum can be written as

$$\begin{aligned}
 H(\lambda, \phi) &= \frac{1}{1 + \mu \sin^2 \phi_2} \left( \frac{1}{2} \lambda_1^2 - \frac{l + \cos \phi_2}{l} \lambda_1 \lambda_2 + \frac{1 + \mu + 2\mu l \cos \phi_2 + \mu l^2}{2\mu l^2} \lambda_2^2 \right) \\
 &\quad + \frac{m_1 g l_1}{E} ((1 + \mu)(1 - \cos \phi_1) + \mu l(1 - \cos(\phi_1 + \phi_2))),
 \end{aligned}$$

where  $\mu = m_2/m_1$ ,  $l = l_2/l_1$  are parameters,  $g$  is the gravitational constant and  $E$  is the energy. Although the system is physically simple, the Hamiltonian is quite complicated. But already very simple topological arguments (cf. Sec. 45 of [7]) indicate intrinsic complexity of this system.

Now I pass to conservative systems with quadratic nonlinearities.

e) *Orszag system* ([63]). This is a simple system of  $n$  differential equations

$$dx_i/dt = ax_{i+1}x_{i+2} + bx_{i-1}x_{i-2} + cx_{i+1}x_{i-1}$$

where

$$x_{i+n} = x_i, \quad 1 \leq i \leq n.$$

Moreover, one supposes

$$a + b + c = 0 \quad \text{and} \quad abc \neq 0.$$

Let us note that the flow induced by this system preserves Lebesgue measure on  $\mathbb{R}^n$  and that the “energy”  $E$ ,

$$E = \frac{1}{2} \sum_{i=1}^n x_i^2,$$

is a first integral of this system, i.e. the spheres in  $\mathbb{R}^n$  centred at the origin are invariant for this system. Thus the Lebesgue measure on these spheres is also invariant. For example, for  $n = 5$  and  $a = b = 1$  and  $c = -2$ , the system displays numerically a chaotic behaviour. Unfortunately, the paper

under consideration is not sufficiently detailed to give a clear evidence about coexistence. I think that this class of systems is worth a further study.

f) *Euler–Poisson equations of motion of a rigid body with a fixed point in the gravitational field.* This is a system of six equations

$$\begin{aligned} Adp/dt &= (B - C)qr + z\gamma' - y\gamma'', \\ Bdq/dt &= (C - A)rp + x\gamma'' - z\gamma, \\ Cdr/dt &= (A - B)pq + y\gamma - x\gamma', \\ d\gamma/dt &= r\gamma' - q\gamma'', \\ d\gamma'/dt &= p\gamma'' - r\gamma, \\ d\gamma''/dt &= q\gamma - p\gamma', \end{aligned}$$

where  $A > 0$ ,  $B > 0$ ,  $C > 0$ ,  $x, y$  and  $z$  are real parameters, and  $(p, q, r, \gamma, \gamma', \gamma'') \in \mathbb{R}^6$ . In what follows we will completely avoid the discussion of the Hamiltonian aspects of these equations (see [33], where one can find many references concerning the rigid body problem).

This system always has three first integrals:

$$\begin{aligned} E &= \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + (x\gamma + y\gamma' + z\gamma''), \\ M &= Ap\gamma + Bq\gamma' + Cr\gamma'', \\ \Gamma &= \gamma^2 + (\gamma')^2 + (\gamma'')^2. \end{aligned}$$

Only in three classical cases of Euler ( $A > 0$ ,  $B > 0$ ,  $C > 0$ ,  $x = y = z = 0$ ) of Lagrange ( $A = B$ ,  $x = y$ ) and of Kovalevskaya ( $A = B = 2C$ ,  $z = 0$ ) a fourth integral is known and the Euler–Poisson equations are integrable. It seems that except for these three cases, in the phase space  $\mathbb{R}^6$  we are always in the presence of TC. Let us consider now the compact invariant manifolds  $E = \text{const}$ ,  $M = \text{const}$ ,  $\Gamma = \text{const} > 0$ . It seems that except the above three cases as well as the so-called Goryachev–Chaplygin case ( $A = B = 4C$ ,  $z = 0$ ,  $M = 0$ ), we are always in the presence of TC.

In one particular case  $A = 3$ ,  $B = 2$  and  $C = 1$ , the numerical study of this problem was given in [33]. The example studied in [33] corresponds to the following simple mechanical situation. One considers the planar rigid body formed by four material points of mass one at the vertices of the unit square, while there is no mass on the edges of the square and the fixed point is in the middle of an edge.

Let us pass now to systems of hydrodynamical origin.

g) *ABC (Arnold–Beltrami–Childress) flow* (see [26] and Appendix 2 of [7]). The system of three differential equations under consideration looks

very simple:

$$\begin{aligned} dx/dt &= A \sin z + C \cos y, \\ dy/dt &= B \sin x + A \cos z, \quad (x, y, z) \bmod 2\pi, \\ dz/dt &= C \sin y + B \cos x, \end{aligned}$$

where  $A, B, C$  are real numbers and  $(x, y, z)$  belongs to the three-dimensional torus  $T^3$ , i.e. one takes  $x, y$  and  $z \bmod 2\pi$ . The flow induced by this system preserves Lebesgue measure on  $T^3$ . In [26] numerical evidence of the presence of TC is given when  $A \neq 0, B \neq 0$  and  $C \neq 0$ . See also [30] for further numerical investigations of the ABC flow.

h) *Vortex flows*. This is the last class of flows I would like to mention here. Like the Newtonian many-body problem, the problem of many interacting vortices is much more a program of study of infinite richness than a single well defined problem.

In the simplest case of the planar problem of  $n$  interacting vortices, the corresponding equations of motion in the Kirchhoff form (1876) are

$$\begin{aligned} \chi_i dx_i/dt &= \partial H / \partial y_i, \\ \chi_i dy_i/dt &= -\partial H / \partial x_i, \quad 1 \leq i \leq n, \end{aligned}$$

with

$$H = -\frac{1}{2\pi} \sum_{1 \leq i < j \leq n} \chi_i \chi_j \log l_{ij}$$

where  $l_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$  and  $\chi_1, \dots, \chi_n$  are fixed real numbers representing the strength of the vortices  $(x_1, y_1), \dots, (x_n, y_n)$  respectively. Up to  $n = 3$ , the equations of vortex motion are integrable. This is no more true for  $n \geq 4$ , where chaotic motions and the coexistence are numerically observed ([6]). In the beautifully written survey articles of H. Aref [3]–[4] and in [5] one can find a more detailed discussion and references. See also [28] and [29]. If one considers the interacting vortex motion in bounded domains, one can observe numerically TC even for two vortices ([48]).

**7. Concluding remarks.** The reader interested in the topic can find many other interesting references in the collections of reprints [35] and [60]. Other useful references are compiled in [38] and [72].

The increasing number of publications about chaos is one of the landmarks of our epoch. It suffices to look at journals like *Celestial Mechanics*, *Communications in Mathematical Physics*, *Ergodic Theory and Dynamical Systems*, *Journal of Differential Equations*, *Journal of Statistical Physics*, *Physica D*, *Physical Review A*, *Physics Letters A*, *Nonlinearity* and many others. Despite all this, *the coexistence problem remains completely open*.

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**Note added in proof** (April 1991)

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The fundamental paper [85], unfortunately overlooked in the main text, contains among other things a first example of coexistence in the geometrical setting of smooth geodesic flows on the two-sphere. [86] contains a first example of coexistence for smooth natural Hamiltonian systems.

The reprint selection [87], which is a sequel to [35], contains a bibliography of 2244 titles limited basically to the first half of 1989.

Finally, the coexistence was observed numerically by E. Busvelle, R. Kharab and the author in many Hamiltonian systems on the Lie algebra  $\mathfrak{so}(4)$  (see Appendix 2 of [7]).

We introduce the concept of dynamical phase coexistence to provide a simple solution for a long-standing problem in theoretical ecology, the so-called "savanna problem". The challenge is to understand why in savanna ecosystems trees and grasses coexist in a robust way with large spatiotemporal variability. We propose a simple model, a variant of the contact process (CP), which includes two key extra features: varying external (environmental/rainfall) conditions and tree age. The system fluctuates locally between a woodland and a grassland phase, corresponding to the active and absorb J.-M. Strelcyn, The "coexistence problem" for conservative dynamical systems: a review,, *Colloq. Math.* 62 (1991), 62 (1991), 331. [22]. M. Wojtkowski, A model problem with the coexistence of stochastic and integrable behavior,, *Comm. Math. Phys.*, 80 (1981), 453. doi: 10.1007/BF01941656. F. Przytycki, Examples of conservative diffeomorphisms of the two-dimensional torus with coexistence of elliptic and stochastic behaviour,, *Ergod. Th. Dynam. Engineering analysis of complex dynamical system.*  $\dot{x} = f(x_1, x_2, \dots, x_n)$ . N.V.Kuznetsov, G.A.Leonov Hidden attractors: systems without equilibria, attractors coexistence & multistability 7/26. Kalman conjecture (Kalman problem) 1957.  $\dot{x} = Ax + b\dot{f}(f)$ ,  $\dot{f} = c\hat{x}$ ,  $\dot{f}(0) = 0$ ,  $0 < \dot{f}(f) < k$ :  $x(t, x_0) \rightarrow 0$  as  $t \rightarrow \infty$ ? J.C. Sprott, A conservative dynamical system with a strange attractor, *Physics Letters A*, 2014. V.-T. Pham, F. Rahma, M. Frasca, L. Fortuna, Dynamics and synchronization of a novel. hyperchaotic system without equilibrium, *Int.*