Monte-Carlo and Quasi-Monte-Carlo Methods for Numerical Integration

HENRI FAURE

Institut de Mathématiques de Luminy, U.P.R. 9016 CNRS 163 avenue de Luminy, case 907, F-13288 Marseille Cedex 09 France

1 INTRODUCTION

We consider the problem of numerical integration in dimension s, with eventually large s; the usual rules need a very huge number of nodes with increasing dimension to obtain some accuracy, say an error bound less than 10^{-2}; this phenomenon is called "the curse of dimensionality"; to overcome it, two kind of methods have been developped: the so-called Monte-Carlo and Quasi-Monte-Carlo methods. Very good and up-to-date monographs on the subject exist; our purpose in the present survey is only to present the basic constructions in the two approaches, with a special insight on the second one which performs better for numerical integration and sets the trend with randomized hybridations . To avoid technicalities, we restrict ourselves to the integral domain $I^s = [0,1]^s$ and to the more commonly used sets of nodes which are already implemented in computers routines; we also leave out the so-called Lattice methods to keep an appropriate length to this proceeding paper. The reader interested who should like to go further in the subject should consult the monographs of H.Niederreiter [N], S.Tezuka [T], M.Drmota-R.Tichy [DT], B.L.Fox[F], J.Matoušek [M] (in chronological order) and also of J.E.Gentle [G] for MC Methods and I.H.Sloan-S.Joe [SJ] for Lattice methods; the collective Springer Lecture notes in Statistics 138 [HL] and the proceedings of the three Conferences MCQMC published by Springer (Lectures Notes in Statistics 106 [NSh], 127 [NHLZ], and the book published this year [NSp]) present the main contributions during the five last years; good references on the Number Theory background for Irregularities of Distribution are the books of J.Beck-W.W.L.Chen [BC] and L.Kuipers-H.Niederreiter [KN].

The goal is to obtain a numerical estimate for $\int_{I^s} f$, with the integrand f in $L^1(I^s)$, for various purposes such that computational Physics, numerical solutions of integro-differential equations (Boltzmann, particle transport), numerical Probabilities, Finance. The approximation is obtained by means of

$$\frac{1}{N}\sum_{n=0}^{N-1} f(X_n)$$

with a well chosen point set $P_N = \{X_0, X_1, ..., X_{N-1}\}$; the choice of the point set

P_N determines the name of the method:

In Monte-Carlo methods, the points are chosen 'randomly' (in the sense of the random functions of computers); the procedure converges almost surely and there is a probabilistic error bound of the form $O\big(\frac{1}{\sqrt{N}}\big)$.

In Quasi Monte-Carlo methods, the points come from deterministic multidimensional sequences with very low irregularities of distribution; there is a deterministic error bound in terms of both the quality of the distribution (by means of the discrepancy $D(P_N)$, see 3.1 for the definition) and the regularity of the integrand (by means of the total variation $V(f)$ of f), the famous Koksma-Hlawka theorem:

$$\Big| \int_{I^s} f - \frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \Big| \leq V(f) \frac{D(P_N)}{N}.$$

One uses the term of "low discrepancy point sets" for point sets P_N satisfying the property: $D(P_N) = O\big((LogN)^{s-1}\big)$.

The second paragraph presents the classical point sets for Monte-Carlo methods, the so-called pseudo-random numbers (PRN) when $s = 1$ or vectors (PRV), with a short analysis of their properties; the third paragraph is devoted to the construction of the simplest multi-dimensional sequences with low irregularities of distribution which produce the low discrepancy point sets for quasi-Monte-Carlo methods, named quasi-random numbers (QRN) and quasi-random vectors (QRV).

2 POINT SETS FOR MONTE-CARLO METHODS

The classical generators produce uniform pseudo-random numbers; the algorithms for the generation of nonuniform random numbers are usually obtained by transformations of uniform one's; to obtain vectors, usually ones takes successive PRN for the different coordinates of the vectors; there are some constructions to generate directly PRV, but this subject is not much developed (see the chapter 10 of [N] for more information).

2.1 The story begins with the linear congruential pseudo-random numbers (LCPRN) proposed by D.H.Lehmer in 1949: given a large integer M, integers a and c with $1 \leq a < M$ and $0 \leq c < M$ and an initial value y_0, he generates the sequence $y_1, y_2, ...$ by the recursion formula for $n = 1, 2, \ldots$:

$$y_n = a y_{n-1} + c \bmod M;$$

then, setting $u_n = y_n/M$ (the so-called normalization), he gets the desired LCPRN in $[0, 1[$.

When $c \neq 0$, an argument of Number Theory shows the LCPRN has maximal period M if and only if $\gcd(c, M) = 1$, $a = 1 \bmod p$ for every prime p dividing M and $a = 1 \bmod 4$ if M is a multiple of 4.

When $c = 0$, the LCPRN has a maximal period if and only if $\gcd(y_0, M) = 1$ and a is a primitive element modulo M; in this case, the generators are also called multiplicative congruential generators.

Usual values for M are 2^{32}, 2^{31} or $2^{31} - 1$, sometimes 2^{48}; for instance the popular RANDU (IBM scientific library subroutine) in the 1960's was obtained with $a = 65539$, $c = 0$ and $M = 2^{31}$; but this generator appeared later to be very bad (see [T]).

The natural generalizations are the multiple recursive generators (MRG) defined by

$$y_n = a_{r-1}y_{n-1} + \cdots + a_0 y_{n-r} \bmod M$$

for $n = r + 1, r + 2, \ldots$ where the initial values $y_1, \ldots, y_r$ are not all 0 and the coefficients a_i are chosen in order to give a very large period, much longer than with the classical LCPRN; these MRG include the so-called lagged Fibonacci generators; the normalization to obtain numbers in $[0, 1[$ is the same as above.

2.2 Beyond linear congruential generators, the general nonlinear congruential method consists in generating the sequence (y_n) by a congruential recursion of arbitrary order with arbitrary (nonlinear) function; first order recursion ($y_n = f(y_{n-1}) \bmod M$) is the most common, especially with $f(x) = ax^{-1} + b$, where x^{-1} is the inverse of x mod M (0 if $x = 0$); such generators are called inversive congruential generators and give ICPRN; usually M is either a prime or a power of two; the normalization, to obtain numbers between 0 and 1, is still the same: $u_n = y_n/M$.

2.3 As pointed out by H.Niederreiter ([N], chapter 9), the preceding method to normalize the recurring sequence is not quite satisfactory; to improve it, the basic idea is to employ small primes p for the modulus and to take sections of the recurring sequence as sets of digits to be used for the computation of the pseudo-random numbers in $[0, 1[$. Two methods based on this idea are of particular interest: the digital multistep method due to R.C.Tausworthe and the generalized feedback shift register method (GFSR) due to T.G.Lewis and W.H.Payne.

In the digital multistep method the recurring sequence is a multiple recursive generator of order r (see 2.1) above) in which the coefficients a_i generate a primitive polynomial over $\mathbf{F}_p$; the normalization to get the sequence (u_n) is obtained by the following formula, where m is an integer between 2 and r:

$$u_n = \sum_{j=1}^{m} y_{mn+j-1} p^{-j}.$$

In the GFSR method, with the same recurring sequence, the normalization results from the formula:

$$u_n = \sum_{j=1}^{m} y_{n+h_j} p^{-j}$$

where $m \geq 2$ and integers $h_1, \ldots, h_m$ are non negative.

Further constructions use combined generators by the XOR operation (P.L' Ecuyer and S.Tezuka) and give some of the best PRN presently used for high-precision Monte-Carlo calculations.

2.4 The assessment of randomness is based on requirements of both theoretical and empirical nature. Theoretical tests give structural information on the distribution properties of the entire period of sequences (spectral test) or provide upper bounds for the deviation from uniform distribution of a part of the sequence (discrepancy test); with long period sequences such tests become unsufficient to look at a part of the period or to give discrepancy upper bounds. So, empirical tests of statistical nature are the last resort in practice; since D.E.Knuth the battery of statistical tests has been enriched by several authors, but seems still unsufficient, according to the capacities of computers for large-scale simulations; among the descriptions given in the books [N] and [T], we may quote (alphabetical order): the autocorrelation test, the frequency test, the gap test, the permutation test, the run test, the serial test; see also [G], in particular for nonuniform distributions.

2.5 Regarding our specific goal of numerical integration, Monte-Carlo methods by means of pseudo-random numbers suffer from three deficiencies which are well brought to the fore in the first chapter of [N]: there are only probabilistic error bounds (almost sure results only), the regularity of the integrand is not reflected and the generation of random samples is a difficult task. As we have seen in paragraph 1, such deficiencies do not persist with quasi-Monte-Carlo methods due to the Koksma-Hlwaka theorem and to the exitence of low-discrepancy sequences; such sequences are constructed in the next paragraph.

3 POINT SETS FOR QUASI-MONTE-CARLO METHODS

3.1 First we need to precise the concept of Discrepancy.

For a point set P_N of N points, $P_N = \{X_0, X_1, ..., X_{N-1}\}$ in $I^s = [0,1]^s$ and a subinterval J of I^s, we define $E(J,N) = A(J,N) - NV(J)$ where $A(J,N)$ is the number of n, $0 \leq n \leq N-1$, with $X_n \in J$ and $V(J)$ is the volume of J; then, following [T], we define the extreme (star) discrepancy of P_N by

$$D(P_N) = \sup_J |E(J,N)|$$

where the supremum is taken over all subintervals J of the form $\Pi_{i=1}^{s}[0, y_i[$.

For an infinite sequence X, we denote by $D(N,X)$ the discrepancy of the point set of its first N points.

A low-discrepancy sequence is an infinite sequence X such that $D(N,X) \leq C_s(LogN)^s$ $(N \geq 2)$, with the constant C_s depending on X and on the dimension s. The next sections are devoted to such sequences.

As to lower bounds for the discrepancy of infinite sequences, except in one dimension, we have 'only' the result of K.F.Roth (1954), slightly improved by R.C.Baker in 1999, for the quadratic (ie L2) discrepancy which gives here $D(N,X) \geq K_s(LogN)^{s/2}$ for infinitely many N, with K_s absolute constant depending only of s; see [BC], [DT] or [N] for more details.

The relation between the discrepancies of infinite sequences and point sets is given by the following inequality:

$$D(P_N) \leq \max_{1 \leq M \leq N}(D(M,X)+1)$$

where the infinite sequence $X=(X_n)$ takes its values in I^{s-1} and where $P_N = \{(\frac{n}{N}, X_n)\ ;\ 0 \leq n \leq N-1\}$ is the corresponding point set of I^s. Therefore, to obtain low-discrepancy points sets in arbitary dimension s, it suffices to construct low-discrepancy sequences in dimension $s-1$. So in the following, we restrict ourselves to infinite sequences.

3.2 The first family is that of van der Corput-Halton.

Let $b \geq 2$ be an integer; for n non-negative integer, let $n = \sum_{r=1}^{\infty} a_r(n) b^{r-1}$ be the b-adic expansion of n; then $\Phi_b(n) = \sum_{r=1}^{\infty} a_r(n) b^{-r}$ is the one-dimensional van der Corput sequence to base b (Φ_b is called the radical inverse function). This sequence is the basic ingredient of all other constructions in arbitrary dimensions.

The first one is that of J.H.Halton (1960): Choose s bases, $b_1, \ldots, b_s$, such that $\mathrm{gdc}(b_i, b_j) = 1$ if $i \neq j$; then $\big(\Phi_{b_1}, ..., \Phi_{b_s}\big)$ is a low-discrepancy sequence (JH) with

$$D(N, JH) \leq \prod_{i=1}^{s} \frac{b_i - 1}{2Log\ b_i} (Log\ N)^s + O\big((Log\ N)^{s-1}\big).$$

The main argument to prove this property is the Chinese remainder theorem.

Recently, this upper bound has been drastically improved by E.Atanassov (divided by $s!$), who constructs a variant of Halton sequences with even better estimate (paper within publication).

3.3 To introduce the second family, which is much larger with many developments, we need to recall the general construction principle initiated in base 2 by I.M.Sobol' (1967), in prime base b by H.Faure (1982) and fully generalized by H.Niederreiter (1987-88), with further improvements by S.Tezuka (1993) and again by H.Niederreiter with C.P.Xing (1995). We begin with some basic and necessary definitions.

3.3.1 An elementary interval in base b ($b \geq 2$) is a sub-interval of I^s of the form

$$E = \prod_{i=1}^{s} \big[a_i b^{-d_i},\ (a_i+1)b^{-d_i}\big[\quad \text{with} \quad 0 \le a_i < b^{d_i} \quad \text{and} \quad d_i \ge 0.$$

A (t, m, s)-net in base b is a point set of b^m points in I^s such that $A(E, b^m) = b^t$ for every elementary interval E in base b with $V(E) = b^{t-m}$ $\quad (0 \le t \le m)$.

A (t, s)-sequence in base b is an infinite sequence in I^s such that for all $k \ge 0$ and $m > t$, the point set $\{X_{kb^m}, ..., X_{(k+1)b^m-1}\}$ is a (t, m, s)-net.

The main theorem on (t, s)-sequences (HN) asserts they are low-discrepancy sequences, with the upper bound:

$$D(N, HN) \le \frac{b^t}{s!}\Big(\frac{b-1}{2Log\ b}\Big)^s (Log\ N)^s + O\big((Log\ N)^{s-1}\big).$$

3.3.2 The general construction principle of (t, s)-sequences, introduced by H.Niederreiter in 1987 and slightly modified by S.Tezuka in 1993, is the following:

(i) R is a commutative ring with $card\ R = b$, $B = \{0, 1, ..., b-1\}$,

(ii) for $r \ge 1$ integer, the ψ_r are bijections from B to R with $\psi_r(0) = 0$ for all sufficiently large r,

(iii) for i, j integers such that $1 \le i \le s$ and $j \ge 1$, the $\lambda_{i,j}$ are bijections from R to B with $\lambda_{i,j}(0) = 0$ for all sufficiently large j and all i,

(iv) for integers i, j, r as above, the $C_{j,r}^{(i)}$ are elements of R.

If $n = \sum_{r=1}^{\infty} a_r(n) b^{r-1}$ is the b-adic expansion of $n \ge 0$, we set $X_n^{(i)} = \sum_{j=1}^{+\infty} x_{n,j}^{(i)} b^{-j}$ where $x_{n,j}^{(i)} = \lambda_{i,j}(\sum_{r=1}^{\infty} C_{j,r}^{(i)} \psi_r(a_r(n)))$.

Then, the sequence $X = (X_n)$, with coordinates $X_n^{(i)}$ as above, is a (t, s)-sequence in base b if the coefficients $C_{j,r}^{(i)}$ are well chosen.

The matrices $C^{(i)} = (C_{j,r}^{(i)})_{j \ge 1, r \ge 1}$ are called the generator matrices of X.

The problem is therefore to choose correctly these matrices; historically, the first constructions are due to I.M. Sobol' in 1967 (b=2 and $t \ge 0$) and H.Faure in 1982 (b prime and $t = 0$).

Many authors use the term of digital (t, s)-sequences for (t, s)-sequences constructed by the preceding principle; in the sequel, we omit the adjective, all the examples being of such kind.

3.3.3 Truncation in base b of a sequence $X = (X_n)_{n \ge 0}$ in I^s:

If the coordinate of index i of the sequence is written $X_n^{(i)} = \sum_{j=1}^{+\infty} x_{n,j}^{(i)}\ b^{-j}$, then the

coordinate-wise $m-$digit truncation in base b is defined by: $\big[X\big]_m = \big(\big[X_n\big]_m\big)_{n\geq 0}$ with $\big[X_n^{(i)}\big]_m = \sum_{j=1}^{m} x_{n,j}^{(i)}\ b^{-j}$.

The truncation has been introduced by S.Tezuka (1993) to allow generator matrices with $C_{j,r}^{(i)} \neq 0$ for infinitely many j (instead of $C_{j,r}^{(i)} = 0$ for all sufficiently large j, in the initial principle of H.Niederreiter).

In the next sections, we introduce the simplest constructions of (t, s)-sequences by giving the generator matrices producing these sequences, according to the general principle above. All point sets for quasi-Monte-Carlo methods resulting from these constructions have been implemented in computers routines and are widely used.

3.4 Constructions of $(0, s)$-sequences in base b.

3.4.1 By the author in 1982: the generator matrices $C^{(i)}$ are powers of the Pascal matrix P modulo b with $b \geq s$, b prime:

$$C^{(i)} = P^{i-1} = \Big(\binom{r}{j}(i-1)^{r-j}\Big)_{r\geq 0, j\geq 0}\ mod\ b \quad (1 \leq i \leq s).$$

In other words, the sequences $(\Phi_b, P\Phi_b, P^2\Phi_b \ldots, P^{s-1}\Phi_b)$ are low-discrepancy sequences (HF), with the upper bound:

$$D(N, HF) \leq \frac{1}{s!}\Big(\frac{b-1}{2Log\ b}\Big)^s (Log\ N)^s + O\big((Log\ N)^{s-1}\big)$$

($t = 0$ in the upper bound given in section 3.3).

The main argument to prove these sequences are $(0, s)$-sequences in base b is a theorem on determinants (generalized van der Monde).

3.4.2 By H.Niederreiter in 1987: the restriction b prime in the preceding construction is removed by computations in the field $\mathbf{F}_b$ with b prime power instead of the field $\mathbf{Z}_b$ when b is prime; better $(0, s)$-sequences are obtained in some dimensions s (for instance, 8 and 9 for $2 \leq s \leq 20$).

3.4.3 By S.Tezuka in 1994: generalization of Faure sequences where the generator matrices are the matrices $C^{(i)} = A^{(i)}P^{i-1}$ with $A^{(i)}$ arbitrary non-singular lower triangular matrices over $\mathbf{F}_b$ and P the Pascal matrix.

Again, by computation of determinants (but see also 3.5.3 below), it is shown the truncated versions of these sequences are $(0, s)$-sequences, with the same upper bound as above.

Well-performing implementations of these sequences with the matrices $A^{(i)}$ randomly chosen are widely used by practitioners in Finance under the name GFaure; an other implementation with empirical carefully chosen matrices $A^{(i)}$ has been produced (Finder software by Papageorgiou and Traub, Columbia University).

A new idea of generalization of Faure sequences has been proposed by the author last year; work in progress, in collaboration with S.Tezuka.

3.5 Constructions of (t, s)-sequences in base b.

3.5.1 By I.M.Sobol' in 1967: the generator matrices are obtained by monocyclic operators over the field $\mathbf{F}_2$; they give (t, s)-sequences in base 2 which appear, 20 years after, to be a particular case of the following constructions.

3.5.2 By H.Niederreiter in 1987-88: according to the formal series framework he introduced in 1988, Niederreiter has developed a wide field of investigations still in progress; we restrict ourselves here to the first constructions which are implemented in computers routines. He got further developments, using algebraic geometry and inducing a new breakthrough in the domain, since 1995, in collaboration with C.P.Xing.

For the finite field $\mathbf{F}_b$, let $G = \mathbf{F}_b((x^{-1}))$ be the field of formal Laurent series $S = \sum_{r=w}^{\infty} a_r x^{-r}$ in x^{-1}, with $a_r \in \mathbf{F}_b$ and w an arbitrary integer.

Let $p_1, p_2, ..., p_s \in \mathbf{F}_b[x]$ be s pairwise coprime polynomials over $\mathbf{F}_b$, with deg $p_i = e_i \geq 1$ and let $g_{i,j} \in \mathbf{F}_b[x]$ be polynomials such that $\gcd(p_i, g_{i,j}) = 1$, with $1 \leq i \leq s$ and $j \geq 1$.

Then the expansion, in which $0 \leq k < e_i$ and $w \leq 0$,

$$\frac{x^k g_{i,j}(x)}{(p_i(x))^j} = \sum_{r=w}^{\infty} a^{(i)}(j, k, r) x^{-r}$$

gives the generator matrices $C^{(i)}$ of the general principle by the following formula:

$$C^{(i)}_{j,r} = a^{(i)}(q + 1, u, r) \ (j \geq 1, \ r \geq 1 \text{ and } j - 1 = qe_i + u \text{ with } 0 \leq u < e_i).$$

The truncated versions of the associated sequences are (t, s)-sequences with $t = \sum_{i=1}^{s} (e_i - 1)$, that is, low-discrepancy sequences with the upper bound of section 3.3.

The proof is based on linear Algebra and polynomial Arithmetic which avoid using computations of determinants and permit the generalization of previous constructions.

Remarks:

In practice, the polynomials are taken in the sequence of irreducible polynomials with increasing degrees, in order to get the smallest value for the parameter t; in some dimensions (for instance 4,14 for $2 \leq s \leq 20$) the estimates are better than with $(0, s)$-sequences.

The last developments by H.Niederreiter and C.P.Xing, which improve the quality of the parameters t and give in some sense the best one's, are still not implemented (difficult task, with several implementation problems to overcome).

The Sobol' sequences correspond to primitive polynomials over $\mathbf{F}_2$ when considered in the formal series framework.

The Faure sequences correspond to polynomials p_i of degree one with b prime.

3.5.3 By S.Tezuka in 1993, with polynomial arithmetic analogues of Halton sequences:

Let $v_n(x) = \sum_{i=0}^{m} n_i x^i$ where $n_i = \psi_{i+1}(a_{i+1}(n))$ and $m = [log_b n)]$, with the notations of the general construction principle of (t, s)-sequences (3.3).

For $p \in \mathbf{F}_b[x]$, with $\deg p = e \leq 1$, write $v_n = \sum_{j=0}^{k} r_j p^j$ with $r_j = [\frac{v_n}{p^j}]$ and k such that $ke \leq m < (k+1)e$. Then

$$\Psi_p(v_n) = \sum_{j=0}^{k} r_j p^{-j-1}$$

is the polynomial version of the radical inverse function (see 3.2 for the integer's one) and the corresponding analogue of Halton sequence is

$$(\sigma_1(\Psi_{p_1}(v_n)), ..., \sigma_s(\Psi_{p_s}(v_n))) \ ,$$

where $p_1, ..., p_s$ are pairwise coprime polynomials and where σ_h is defined by

$$\sigma_h(\sum_{r=w}^{\infty} a_r x^{-r}) = \sum_{r=w}^{\infty} \lambda_{h,r}(a_r) b^{-r}.$$

The truncated versions of these analogues of Halton sequences are (t, s)-sequences in base b with $t = \sum_{i=1}^{s} (e_i - 1)$ and with the upper bound of section 3.3.

3.6 Scrambled sequences.

The idea of using permutations on the digits $x_{n,j}^{(i)}$ (see the general construction principle in 3.3) to reduce the discrepancy was first used by applied mathematicians (T.Warnock, 1972, E.Braaten and G.Weller, 1979) and by the author for detailed studies in one dimension (1981, 1990-93). Then, a systematic investigation of randomized nets by uniform random permutations has been developped, from a statistical point of view, by A.Owen (1994-97) in three papers at least, the first one in [NSh]; further works by B.L.Fox, F.Hickernell and H.S.Hong, J.Matoušek, H.Morohosi and M.Fushimi, bring a new light on these hybrid methods mixing

pseudo-random and quasi-random techniques; see also the GFaure sequences of S.Tezuka in 3.4.3 which, in high dimensional ($s > 300$) simulation for Finance problems, give very efficient results, much faster than Monte-Carlo; nevertheless, these results are empirical and few theoretical knowledge is available to explain them; currently, this phenomenon attracts more and more attention from both pure and applied mathematicians and should be cleared up in the future.

3.7 Constructions of (t, m, s)-nets.

As pointed out at the end of section 3.1, a good way to obtain point sets for QMC methods is to construct (t, s)-sequences; but it is not the only one and it is even possible to get better point sets (that is with lower discrepancy) in specific cases, especially with small bases and small dimensions; a lot of researchers have worked on this topic and large tables of (t, m, s)-nets are available with calculations of the quality parameter t; one of the best set of such tables is known as 'Salzburg tables', still improved and with constant up-dating on a web site (see www.mat.sbg.ac.at/schmidw); the last issue of the MCQMC Conferences [NSp] contains two papers of H.Niederreiter and W.C.Schmid giving the status of the work on July 98; but since, new progress have been done, at least by G.Pirsic and W.C.Schmid (to appear).

References

Almost all references on the subject can be found in the following list of books; we have restricted ourselves to that list to avoid a too long enumeration and inevitable omissions.

[BC] J.Beck and W.W.L.Chen. Irregularities of Distribution. Cambridge University Press, Cambridge, 1987.

[DT] M.Drmota and R.Tichy. Sequences, Discrepancies and Applications. Lectures Notes in Mathematics 1651, Springer, 1997 (2000 references).

[F] B.L.Fox. Strategies for Quasi-Monte Carlo. Kluwer, 1999.

[G] J.E.Gentle. Random Number Generation and Monte Carlo Methods. Statistics and Computing, Springer,1998.

[HL] P.Hellekalek and G.Larcher (Editors). Random and Quasi-Random Point Sets. Lectures notes in Statistics 138, Springer, 1998.

[KN] L.Kuipers and H.Niederreiter. Uniform Distribution of Sequences. Wiley-Interscience Publishers, 1974.

[M] J.Matoušek. Geometric discrepancy. Springer, 1999.

[N] H.Niederreiter. Random Number Generation and quasi-Monte Carlo Methods. SIAM Conf.Ser.Appl. Vol.63, Philadelphia, 1992.

[NHLZ] H.Niederreiter, P.Hellekalek, G.Larcher and P.Zinterhof (Editors).

MCQMC'96. Lectures notes in Statistics 127, Springer, 1998.

[NSh] H.Niederreiter and P.J.S.Shiue (Editors). MCQMC in Scientific Computing. Lectures notes in Statistics 106, Springer, 1995.

[NSp] H.Niederreiter and J.Spanier (Editors). MCQMC'98. Springer, 2000.

[SJ] I.H.Sloan and S.Joe. Lattice Methods for Multiple Integration. Oxford University Press, Oxford, 1994.

[T] S.Tezuka. Uniform random numbers: Theory and practice. Kluwer Academic Publishers, Dordrecht, 1995.

Quasi-Monte Carlo methods use sequences that are deterministic instead of random. These sequences improve convergence and give rise to deterministic error bounds. The method is explained and illustrated with several examples.Â Such sequences are often more efficient than standard Monte Carlo in evaluating high dimensional integrals when the integrand is sufficiently regular, which is the case for many ap- plications in finance. Applications of low discrepancy sequences to financial problems are studied by Boyle, Broadie and Glasserman (1997), Cashisch, Morokoff and Owen (1997), Joy, Boyle and Tan (1996), Ninomiya and Tezuka (1996), Tan and Boyle (2000) and Paskov and Traub (1995) [10,11,25,34,36,40,41,42]. Introduction to numerical integration using Monte Carlo and quasi-Monte Carlo techniques. Transcript. 1. Monte Carlo and quasi-Monte Carlo Integration John D. Cook M. D. Anderson Cancer Center July 24, 2002 2. Trapezoid rule in one dimensionError bound proportional to product of Step size squared Second derivative of integrand N = number of function evaluations Step size h = N-1 Error proportional to N-2 3. Simpsons rule in one dimensionsError bound proportional to product of.Â Contrasting proportionality Classical methods improve with smooth integrands Monte Carlo doesnt depend on differentiability at all, but improves with overall flatness. 12. Good MC, bad trapezoid1 0.8 0.6 0.4 0.21.5 2 2.5 3 13. Good trapeziod, bad MC8 6 4 2 -3 -2 -1 1 2 3 14.